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A Theory for the Small Unsymmetric Deformations of Cylindrical Shells

EDWARD L. REISS

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Edward L. Reiss

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A Theory For The Small Unsymmetric Deformations of Cylindrical Shells

1. Introduction

In reference 1 approximate theories to determine the rotationally symmetric deformations of "thick" cylindrical shells were developed in a systematic manner from the exact three dimensional theory of linear elasticity for an isotropic and homogeneous material (hereafter referred to as the exact theory). These approximate theories were obtained by a method that is in a sense, a generalization of the boundary layer expansion technique used by Friedrichs[2] and later Friedrichs and Dressler[3] in a study of the bonding of plates.* However the scaling and expansion techniques differ somewhat from those in reference and 3.

In this paper, the method presented in reference 1 is extended to include unsymmetric deformations of cylindrical shells. Similar scaling and expansion procedures are employed and the present results specialize to those of reference 1 for rotationally symmetric deformations. The basic element in the approximation is a "thin" shell theory. The "thick" shell theories are then obtained as "corrections" to the thin shell theory. We find that the so called Donnell theory of cylindrical shells[5], which appears naturally as a consequence of our expansion procedures, is the appropriate thin shell theory. A summary of the thick shell theory is given in Section 6.

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*In this connection see also reference 4.

2. Formulation.

In terms of the cylindrical coordinate system, (r, θ, x), a cylindrical shell of uniform thickness $2h$, radius R and length L is defined as an elastic body bounded by the coaxial surfaces $r = R \pm h$ and the planes $x = 0$ and $x = L$. The inner and outer surfaces of the cylinder, $r = R - h$ and $r = R + h$, respectively, are subjected only to normal pressures. The edges of the cylinder formed by the annular regions $x = 0, L, R - h \leq r \leq R + h$, are submitted to normal and shear forces. The applied forces are arbitrary provided they form a system in equilibrium.

Our purpose is to obtain, in a systematic manner from the exact theory, two dimensional or "shell" theories which approximate the deformations of cylindrical bodies with "small" values of h/R. It is convenient to introduce the dimensionless variables:

$$(1) \qquad \xi = \frac{x}{R\varepsilon^a} = \frac{x}{\sqrt{Rh}}, \quad \phi = \frac{\theta}{\varepsilon^a} = \frac{\theta}{(h/R)^{1/2}}, \quad \zeta = \frac{r-R}{R\varepsilon^{2a}} = \frac{r-R}{h},$$

where, a, is a positive integer and ε is the dimensionless parameter,

$$(2) \qquad \varepsilon = \left(\frac{h}{R}\right)^{1/2a} .$$

The inner and outer surfaces are thus given by $\zeta = \pm 1$, respectively, while the ends of the cylinder correspond to $\xi = 0, L'$ where, $L' = \frac{L}{R\varepsilon^a} = \frac{L}{\sqrt{Rh}}$. A more general scaling procedure than (1) and (2) may be used. However, this leads to "unreasonable" shell theories which are discussed in reference 1

for the special case of rotationally symmetric deformations. Attention in this paper is therefore restricted to the scaling in (1) and (2). The integer, a, provides some flexibility in the choice of the parameter, ε. For simplicity of presentation we set a = 1 and hence,*

$$\varepsilon = (h/R)^{1/2} .$$

Dimensionless stress and displacement components are defined by dividing the physical stresses by Young's modulus and the physical displacements by R. Using these dimensionless quantities and the variables (1) and (2), the equations of the exact theory neglecting body forces are, with an obvious notation:

equilibrium equations,

$$\sigma_{z,\zeta} + \varepsilon\sigma_{zx,\xi} + g(\zeta)[\sigma_z - \sigma_\theta) + \varepsilon^{-1}\sigma_{z\theta,\phi}] = 0 \quad ,$$

(3) $$\sigma_{zx,\zeta} + \varepsilon\sigma_{x,\xi} + g(\zeta)[\sigma_{zx} + \varepsilon^{-1}\sigma_{x\theta,\phi}] = 0 \; ,$$

$$\sigma_{z\theta,\zeta} + \varepsilon\sigma_{x\theta,\xi} + g(\zeta)[2\sigma_{z\theta} + \varepsilon^{-1}\sigma_{\theta,\phi}] = 0 \; ;$$

stress displacement relations (Hooke's Law),

*In principle, no difficulty is encountered if, a, is an arbitrary integer. However the calculations become more cumbersome; see reference 1.

$$
\begin{aligned}
& u_{,\xi} = \varepsilon[\sigma_x - \nu(\sigma_\theta + \sigma_z)], \\
& v_{,\phi} + \varepsilon w = \varepsilon(1 + \varepsilon^2\zeta)[\sigma_\theta - \nu(\sigma_x + \sigma_z)], \\
& u_{,\phi} + (1 + \varepsilon^2\zeta)v_{,\xi} = \varepsilon(1 + \varepsilon^2\zeta)2(1 + \nu)\sigma_{x\theta}, \\
(4) \qquad & w_{,\zeta} = \varepsilon^2[\sigma_z - \nu(\sigma_x + \sigma_\theta)], \\
& u_{,\zeta} + \varepsilon w_{,\xi} = \varepsilon^2 2(1 + \nu)\sigma_{zx}, \\
& v_{,\zeta} - g(\zeta)[v - \varepsilon^{-1}w_{,\phi}] = \varepsilon^2 2(1 + \nu)\sigma_{z\theta};
\end{aligned}
$$

Compatibility Equations,

$$
\begin{aligned}
& \Delta\sigma_x + \varepsilon^2\Omega_{,\xi\xi} = 0, \\
& \Delta\sigma_z + \Omega_{,\zeta\zeta} - 2g^2(\zeta)(\sigma_z - \sigma_\theta + 2\varepsilon^{-1}\sigma_{z\theta,\phi}) = 0, \\
& \Delta\sigma_\theta + g(\zeta)\Omega_{,\zeta} + 2g^2(\zeta)(\sigma_z - \sigma_\theta + 2\varepsilon^{-1}\sigma_{z\theta,\phi} \\
& \qquad\qquad + \frac{\varepsilon^{-2}}{2}\Omega_{,\phi\phi}) = 0, \\
(5) \quad & \Delta\sigma_{zx} + \varepsilon\Omega_{,\zeta\xi} - g^2(\zeta)(\sigma_{zx} + 2\varepsilon^{-1}\sigma_{x\theta,\phi}) = 0, \\
& \Delta\sigma_{z\theta} + \varepsilon^{-1}g(\zeta)\Omega_{,\zeta\phi} \\
& \qquad + 2g^2(\zeta)[-2\sigma_{z\theta} + \varepsilon^{-1}\frac{\partial}{\partial\phi}(\sigma_z - \sigma_\theta - \tfrac{1}{2}\Omega)] = 0, \\
& \Delta\sigma_{x\theta} + g(\zeta)\Omega_{,\xi\phi} + g^2(\zeta)(-\sigma_{x\theta} + \varepsilon^{-1}\sigma_{zx,\phi}) = 0,
\end{aligned}
$$

where ν is Poisson's ratio and,

$$
g(\zeta) = \frac{\varepsilon^2}{1+\varepsilon^2\zeta}, \qquad \Omega = \frac{1}{1+\nu}[\sigma_x + \sigma_\theta + \sigma_z],
$$

$$
\Delta \equiv \frac{\partial^2}{\partial\zeta^2} + \varepsilon^2\frac{\partial^2}{\partial\xi^2} + g(\zeta)\frac{\partial}{\partial\zeta} + \varepsilon^{-2}g^2(\zeta)\frac{\partial^2}{\partial\phi^2}.
$$

In (3.5) we have employed the conventional notation that an independent variable appearing as a subscript following a comma denotes differentiation with respect to that variable. Thus,

$$\sigma_{z\theta,\zeta} = \frac{\partial \sigma_{z\theta}}{\partial \zeta} .$$

A complete formulation of the elastic problem for the cylinder (Formulation A) consists of (3) and (4) and appropriate boundary conditions on the surfaces and ends of the cylinder. A second and equivalent formulation (Formulation B) consists of (3) and (5) with appropriate boundary conditions. These conditions may be obtained by specifying the applied forces in the following manner:

$$\text{(6a)} \quad \begin{cases} \sigma_{zx}(\xi,\phi,\pm 1;\varepsilon) = \sigma_{z\theta}(\xi,\phi,\pm 1;\varepsilon) = 0 , \\ \sigma_{z}(\xi,\phi,1;\varepsilon) = p_{o}(\xi,\phi;\varepsilon), \; \sigma_{z}(\xi,\phi,-1;\varepsilon) = p_{I}(\xi,\phi;\varepsilon); \end{cases}$$

$$\text{(6b)}^{*} \quad \begin{cases} \sigma_{x}(0,\phi,\zeta;\varepsilon) = \bar{\sigma}(\phi,\zeta;\varepsilon), \; \sigma_{zx}(0,\phi,\zeta;\varepsilon) = \bar{\tau}(\phi,\zeta;\varepsilon) , \\ \sigma_{x\theta}(0,\phi,\zeta;\varepsilon) = \bar{t}(\phi,\zeta;\varepsilon) ; \end{cases}$$

and conditions similar to (6b) on $\xi = L$. Here $p_{o}(\xi,\phi;\varepsilon)$ and $p_{I}(\xi,\phi;\varepsilon)$ are, respectively, the applied normal forces on the outer and inner surfaces of the cylinder. $\bar{\sigma}(\phi,\zeta;\varepsilon)$, $\bar{\tau}(\phi,\zeta;\varepsilon)$ and

---

* $\bar{\tau}(\phi, \pm 1;\varepsilon) = \bar{t}(\phi, \pm 1;\varepsilon) = 0$ for continuity.

$\mathfrak{t}(\phi,\zeta;\varepsilon)$ are the forces applied to the ends of the cylinder. We further impose the unessential but convenient restriction that,

$$p_O \neq p_I \, . \tag{7}$$

Formulation A is employed in the following sections to discuss the deformations in the "interior" of the cylinder. To examine the deformations in the "boundary layer"* we use Formulation B in sections 4 and 5.

3. The Interior Problems.

We assume that each component of stress and displacement, indicated by the generic symbol, $\sigma(\xi,\phi,\zeta;\varepsilon)$, can be represented, asymptotically, as a power series in ε:

$$\sigma(\xi,\phi,\zeta;\varepsilon) \sim \sum_{n=0}^{\infty} \sigma^n(\xi,\phi,\zeta)\varepsilon^n \tag{8}$$

where $\sigma^n(\xi,\phi,\zeta) = 0$ if $n < 0$. The functions $\sigma^n(\xi,\phi,\zeta)$ are called the interior stress coefficients or the interior displacement coefficients of order n which ever the case may be. We further assume that the prescribed forces in (6) can be expanded in power series in ε:

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* The term "boundary layer" refers to a narrow volume of the cylinder adjacent to and including the ends where stresses and deflections may change rapidly in the ξ direction. The remaining part of the cylinder is called the "interior".

$$(9.a)\qquad \begin{Bmatrix} p_o(\xi,\phi;\varepsilon) \\ p_I(\xi,\phi;\varepsilon) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} p_o^n(\xi,\phi) \\ p_I^n(\xi,\phi) \end{Bmatrix} \varepsilon^n \quad ;$$

$$(9.b)\qquad \begin{Bmatrix} \bar{\sigma}(\phi,\zeta;\varepsilon) \\ \bar{\tau}(\phi,\zeta;\varepsilon) \\ \bar{t}(\phi,\zeta;\varepsilon) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} \bar{\sigma}^n(\phi,\zeta;\varepsilon) \\ \bar{\tau}^n(\phi,\zeta;\varepsilon) \\ \bar{t}^n(\phi,\zeta;\varepsilon) \end{Bmatrix} \varepsilon^n \quad .$$

Expansions of the form (8) are substituted into the equilibrium equations (3) and the stress displacement relations (4). Coefficients of the same powers of ε are equated yielding a system of differential equations satisfied by the interior stress and displacement coefficients of all orders. The coefficients of ε^n are:

$$(10a)\qquad \sigma^n_{z,\zeta} + \sigma^{n-1}_{zx,\xi} + \sum{}'(\sigma^i_z - \sigma^i_\theta) + \sum{}''\sigma^i_{z\theta} = 0 \quad ,$$

$$(10b)\qquad \sigma^n_{zx,\zeta} + \sigma^{n-1}_{x,\xi} + \sum{}'\sigma^i_{zx} + \sum{}''\sigma^i_{x\theta} = 0 \quad ,$$

$$(10c)\qquad \sigma^n_{z\theta,\zeta} + \sigma^{n-1}_{x\theta,\xi} + 2\sum{}'\sigma^i_{z\theta} + \sum{}''\sigma^i_\theta = 0 \quad ;$$

$$(11a)\qquad u^n_{,\zeta} = \sigma^{n-1}_x - \nu(\sigma^{n-1}_\theta + \sigma^{n-1}_z) \quad ,$$

$$(11b)\qquad v^n_{,\phi} + w^{n-1} = \sigma^{n-1}_\theta - \nu(\sigma^{n-1}_x + \sigma^{n-1}_z)$$

$$+ \zeta[\sigma^{n-3}_\theta - \nu(\sigma^{n-3}_x + \sigma^{n-3}_z)] \quad ,$$

(11c) $$v^n_{,\xi} + \zeta v^{n-2}_{,\xi} + u^n_{,\phi} = 2(1+\nu)[\sigma^{n-1}_{x\Omega} + \zeta\sigma^{n-3}_{x\Omega}] \quad ,$$

(11d) $$w^n_{,\zeta} = \sigma^{n-2}_z - \nu(\sigma^{n-2}_x + \sigma^{n-2}_\Omega) \quad ,$$

(11e) $$u^n_{,\zeta} + w^{n-1}_{,\xi} = 2(1+\nu)\sigma^{n-2}_{zx} \quad ,$$

(11f) $$v^n_{,\zeta} + \zeta v^{n-2}_{,\zeta} - v^{n-2} + w^{n-1}_{,\phi} = 2(1+\nu)[\sigma^{n-2}_{z\Omega} + \zeta\sigma^{n-4}_{z\Omega}] \quad .$$

Here,

$$\left.\begin{aligned} \sum{}' A^i &= \sum_{i+2(j+1)=n}\sum (-1)^j \zeta^j A^i \\ \sum{}'' A^i &= \sum_{i+2j+1=n}\sum (-1)^j \zeta^j A^i_{,\phi} \end{aligned}\right. \qquad i,j \geq 0 \quad .$$

Appropriate boundary conditions on $\zeta = \pm 1$ for the interior stress and displacement coefficients are obtained by substituting (9a) and expansions of the form (8) into (6a) and equating coefficients of like powers of ε. This yields, for the coefficients of ε^n,

(12a) $$\sigma^n_{zx}(\xi,\phi,\pm 1) = \sigma^n_{z\Omega}(\xi,\phi,\pm 1) = 0 \quad ,$$

(12b) $$\left\{\sigma^n_z(\xi,\phi,1),\ \sigma^n_z(\xi,\phi,-1)\right\} = \left\{p^n_O(\xi,\phi),\ p^n_I(\xi,\phi)\right\}$$

Equations to determine each interior coefficient are obtained from (10-12). We first solve (11a,b,c) for σ^n_x, σ^n_Ω and $\sigma^n_{x\Omega}$,

$$\sigma_x^n = \frac{1}{1-\nu^2}[u_{,\xi}^{n+1} + \nu(v_{,\phi}^{n+1} + w^n)]$$

$$+ \frac{\nu}{1-\nu}\sigma_z^n - \frac{\nu}{1-\nu^2}[\sigma_\theta^{n-2} - \nu(\sigma_x^{n-2} + \sigma_z^{n-2})]\zeta \quad ,$$

(13) $$\sigma_\theta^n = \frac{1}{1-\nu^2}[v_{,\phi}^{n+1} + w^n + \nu u_{,\xi}^{n+1}]$$

$$+ \frac{\nu}{1-\nu}\sigma_z^n - \frac{1}{1-\nu^2}[\sigma_\theta^{n-2} - \nu(\sigma_x^{n-2} + \sigma_z^{n-2})]\zeta \quad ,$$

$$\sigma_{x\theta}^n = \frac{1}{2(1+\nu)}[v_{,\xi}^{n+1} + u_{,\phi}^{n+1} + \zeta v_{,\xi}^{n-1}] - \zeta\sigma_{x\theta}^{n-2} \quad .$$

If $n = 0$, it follows from (11), recalling that $\sigma^n = 0$ if $n < 0$, that,

$$u_{,\xi}^o = u_{,\zeta}^o = v_{,\phi}^o = v_{,\zeta}^o = v_{,\xi}^o + u_{,\phi}^o = w_{,\zeta}^o = 0.$$

Therefore,

(14) $$u^o(\xi,\phi,\zeta) = v^o(\xi,\phi,\zeta) = 0, w^o(\xi,\phi,\zeta) = W^o(\xi,\phi) \quad ,$$

where $W^o(\xi,\phi)$ is an arbitrary function to be subsequently determined.* Similarly we find from (10b,c) and (12a) that,

(15a) $$\sigma_{zx}^o(\xi,\phi,\zeta) = \sigma_{z\theta}^o(\xi,\phi,\zeta) = 0 \quad .$$

Using this result in conjunction with (12b) and (7), Eq. (10a) yields,

---

*Here we have assumed that $u^o(\xi,\phi,\zeta)$ is periodic in ϕ and have neglected rigid body deformations.

(15b) $$\sigma_z^n = p_o^n = p_I^n = 0, \quad \text{if } n < 2.$$

Returning to (11d) we see that,

(16) $$w^n(\xi,\phi,\zeta) = W^n(W,\phi) \ , \quad \text{if } n < 2.$$

Employing this last result, (14) and (15a), (11e,f) yield,

$$u^n_{,\zeta} + w^{n-1}_{,\xi} = v^n_{,\zeta} + w^{n-1}_{,\phi} = 0, \qquad \text{if } n = 1,2,$$

which implies that,

(17) $$\left.\begin{aligned} u^n(\xi,\phi,\zeta) &= -W^{n-1}_{,\xi}\zeta + U^n(\xi,\phi) \ , \\ v^n(\xi,\phi,\zeta) &= -W^{n-1}_{,\phi}\zeta + V^n(\xi,\phi), \end{aligned}\right\} \quad \text{if } n = 1,2 \ .$$

Here $U^n(\xi,\phi)$ and $V^n(\xi,\phi)$ are arbitrary functions which are determined later.

Substitution of (15) and (17) into (13) gives the first two stress-displacement relations for σ_x^n, σ_Ω^n and $\sigma_{x\Omega}^n$ in terms of U^1, V^1 and W^o, and U^2, V^2 and W^1:

(18) $$\left.\begin{aligned} \sigma_x^n(\xi,\phi,\zeta) &= S_x^n(\xi,\phi) + \overline{S}_x^n(\xi,\phi)\zeta \ , \\ \sigma_\Omega^n(\xi,\phi,\zeta) &= S_\Omega^n(\xi,\phi) + \overline{S}_\Omega^n(\xi,\phi)\zeta \ , \\ \sigma_{x\Omega}^n(\xi,\phi,\zeta) &= S_{x\Omega}^n(\xi,\phi) + \overline{S}_{x\Omega}^n(\xi,\phi)\zeta \ , \end{aligned}\right\} \quad \text{if } n = 0,1 \ ,$$

where,*

$$
\text{(19a)}\quad
\begin{cases}
S_x^n(\xi,\phi) = \dfrac{1}{1-\nu^2}[U^{n+1}_{,\xi} + \nu(V^{n+1}_{,\phi} + W^n)] \quad , \\[2ex]
S_\theta^n(\xi,\phi) = \dfrac{1}{1-\nu^2}[V^{n+1}_{,\phi} + W^n + \nu U^{n+1}_{,\xi}] \quad , \\[2ex]
S_{x\theta}^n(\xi,\phi) = \dfrac{1}{2(1+\nu)}[V^{n+1}_{,\xi} + U^{n+1}_{,\phi}] \quad ,
\end{cases}
$$

$$
\text{(19b)}\quad
\begin{cases}
\bar{S}_x^n(\xi,\phi) = \dfrac{-1}{1-\nu^2}[W^n_{,\xi\xi} + \nu W^n_{,\phi\phi}] \quad , \\[2ex]
\bar{S}_\theta^n(\xi,\phi) = -\dfrac{1}{1-\nu^2}[W^n_{,\phi\phi} + \nu W^n_{,\xi\xi}] \quad , \\[2ex]
\bar{S}_{x\theta}^n(\xi,\phi) = -\dfrac{1}{1+\nu} W^n_{,\phi\xi} \quad .
\end{cases}
$$

We now obtain two systems of differential equations to determine U^1, V^1 and W^o, and U^2, V^2 and W^1. Substituting (15a) and (18) into (10b,c) with $n = 1$ or 2, integrating with respect to ζ and using the surface conditions (12a) there results,

$$
\text{(20)}\qquad S^n_{x,\xi} + S^n_{x\theta,\phi} = 0, \quad S^n_{x\theta,\xi} + S^n_{\theta,\phi} = 0, \quad \text{if } n = 0,\ 1,
$$

and

$$
\text{(21)}\quad
\left.
\begin{aligned}
\sigma^n_{zx}(\xi,\phi,\zeta) &= \tfrac{1}{2}[\bar{S}^{n-1}_{x,\xi} + \bar{S}^{n-1}_{x\theta,\phi}](1-\zeta^2) \ , \\
\sigma^n_{z\theta}(\xi,\phi,\zeta) &= \tfrac{1}{2}[\bar{S}^{n-1}_{\theta,\phi} + \bar{S}^{n-1}_{x\theta,\xi}](1-\zeta^2) \ ,
\end{aligned}
\right\}\ \text{if } n = 1,2 \ .
$$

---

*The coefficients defined in (19) are proportional to the conventional stress resultants and couples.[1,6,7]

Equations (20) supply four of the required differential equations. Equations (21) are the stress displacement relations for the first two non-vanishing interior transverse shear stress coefficients. The remaining two differential equations and stress displacement relations for the first two non-vanishing σ_z^n are obtained by substituting (15), (18) and (21) into (10a) with n = 2 or 3. Integrating the resulting equation once with respect to ζ and using the surface conditions (12b) we find that,

$$(22) \qquad \bar{S}^n_{x,\xi\xi} + 2\bar{S}^n_{x\theta,\phi\xi} + \bar{S}^n_{\theta,\phi\phi} - 3S^n_\theta = 3(p_I^{n+2} - p_o^{n+2}),$$

$$\text{if } n = 0,1 \quad ,$$

and

$$(23) \qquad \sigma_z^n(\xi,\phi,\zeta) = \tfrac{3}{2}(-S_\theta^{n-2} + p_o^n - p_I^n)(\zeta - \tfrac{\zeta^3}{3}) - \tfrac{1}{2}\bar{S}_\theta^{n-2}(1 - \zeta^2)$$

$$+ S_\theta^{n-2}\zeta + (p_o^n + p_I^n) \ , \quad \text{if } n = 2,3 \ .$$

It is now assumed, with no loss of generality, that both $p_o^2(\xi,\phi)$ and $p_I^2(\xi,\phi)$ do not simultaneously identically vanish. This implies that the normal forces applied to the inner and outer surfaces are of order of magnitude $\varepsilon^2 = \frac{h}{R}$.

To summarize, we have obtained by a systematic expansion procedure relations expressing each of the first two non-vanishing interior stress coefficients in terms of the first two non-vanishing interior displacement coefficients. In addition appropriate differential equations to determine these displacement coefficients have been obtained. These results may be written in

a more familiar form. Substituting (19) into (20) and (22) and setting $n = 0$ gives the differential equations of the "zeroth order interior problem" as:

$$U^1_{,\xi\xi} + \frac{1-\nu}{2} U^1_{,\phi\phi} + \frac{1+\nu}{2} V^1_{,\phi\xi} + \nu W^o_{,\xi} = 0 \quad ,$$

(24) $$V^1_{,\phi\phi} + \frac{1-\nu}{2} V^1_{,\xi\xi} + \frac{1+\nu}{2} U^1_{,\phi\xi} + W^o_{,\phi} = 0 \quad ,$$

$$\triangle^2 W^o + 3(V^1_{,\phi} + W^o + \nu U^1_{,\xi}) = 3(1 - \nu^2)(p_o^2 - p_I^2) \quad ;$$

while from (18), (19), (21) and (23) the stress displacement relations of the zeroth order interior problem are,

(25) $$\begin{cases} \sigma^o_x(\xi,\phi,\zeta) = S^o_x(\xi,\phi) + \overline{S}^o_x(\xi,\phi)\zeta \quad , \\ \sigma^o_\theta(\xi,\phi,\zeta) = S^o_\theta(\xi,\phi) + \overline{S}^o_\theta(\xi,\phi)\zeta \quad , \\ \sigma^o_{x\theta}(\xi,\phi,\zeta) = S^o_{x\theta}(\xi,\phi) + \overline{S}^o_{x\theta}(\xi,\phi)\zeta \quad , \end{cases}$$

(26) $$\begin{cases} \sigma^1_{zx}(\xi,\phi,\zeta) = -\dfrac{1}{2(1-\nu^2)}(W^o_{,\xi\xi\xi} + W^o_{,\phi\phi\xi})(1 - \zeta^2) \quad , \\ \sigma^1_{z\theta}(\xi,\phi,\zeta) = -\dfrac{1}{2(1-\nu^2)}(W^o_{,\xi\xi\phi} + W^o_{,\phi\phi\phi})(1 - \zeta^2) \quad , \\ \sigma^2_z(\xi,\phi,\zeta) = \dfrac{1}{2(1-\nu^2)}[\triangle^2 W^o(\zeta - \zeta^3/3) \end{cases}$$

$$+ (W^o_{,\phi\phi} + \nu W^o_{,\xi\xi})(1 - \zeta^2)$$

$$+ 2(V^1_{,\phi} + W^o + \nu U^1_{,\xi})\zeta] + (p_o^2 + p_I^2) \quad .$$

Here Δ^2 is the two-dimensional biharmonic operator

$$\Delta^2 \equiv \frac{\partial^4}{\partial\xi^4} + 2\,\frac{\partial^4}{\partial\xi^2\partial\phi^2} + \frac{\partial^4}{\partial\phi^4}$$

and the coefficients in (25) are given in (19). The differential equations and stress displacement relations of the "first order interior problem" are similarly obtained from (18) - (23). These expressions will be referred to as Eq. (27). They are identical with (24) - (26) provided that a, 1, is added to the superscript of each term. If the deformations are rotationally symmetric then all the previous results reduce to those of reference 1.

The differential equations (24) and the stress displacement relations (25) are, in our notation, identical with those of the thin shell theory of Donnell.[5] Equations (26), which are not given in the Donnell theory, provide "thick shell corrections" to this thin shell theory. Thus the Donnell equations appear in a natural way as part of the first approximation of the exact theory.*

The differential equations and stress displacement relations for interior problems of order two or greater are obtained, in a similar fashion, from (10 - 13). Because of the complexity of the analysis they are not explicitly shown here. We merely remark that, σ_x^2, σ_θ^2, $\sigma_{x\theta}^2$, u^3 and v^3 are cubics in ζ, while w^2 is quadratic in ζ.

To complete the formulation of the interior problems, boundary conditions appropriate to the differential equations (20) and (22) and (27) are required. These boundary conditions are systematically

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*This does not necessarily imply that other thin shell theories,[5-8] could not serve as first approximations to the exact equations if different expansion procedures or parameters are employed.

derived in sections 4 and 5 from the exact theory. We observe that all the results for the interior problems are obtained without reference to the edge boundary conditions, (6b).

4. Formulation of the Boundary Layer Problem.

The specific dependence of the interior stress coefficients $\sigma_x^n, \sigma_{zx}^n$ and $\sigma_{x\Omega}^n$ on ζ is given by the interior analysis.* However the boundary conditions on the ends of the shell, (6b), imply that these coefficients are arbitrary functions of ζ. Therefore, the expansions (8) cannot satisfy these conditions and if they represent the solution they do so only in a region away from the ends, i.e. (8) are not uniformly valid. To obtain expansions that are valid up to and including the end $\xi = 0$, we introduce a "boundary layer coordinate", η, by "stretching" the variable ξ so that the resulting expansions may represent the solution uniformly:†

$$(28) \qquad \eta = \frac{\xi}{\varepsilon} = \frac{x}{h} \quad .$$

Thus by making ε sufficiently small, every neighborhood of the end in the ξ variable corresponds to an arbitrarily large one in the η variable. The ϕ variable is not stretched. This implies that the rapid variations that occur near the boundary take place only in the ξ direction, i.e., the direction normal to the boundary.

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*For example, see (25) and (26).

†For descriptions of these boundary layer methods see References 1 - 4. In the following we confine our attention to the end $\xi = 0$. An identical analysis is valid for the end $\xi = L'$.

We introduce (28) into formulation B of Section 2 and define the boundary layer stresses indicated by the generic symbol, $f(\eta,\phi,\zeta;\varepsilon)$, as the physical stresses divided by Young's modulus. Each of the boundary layer stresses is assumed to be representable by a power series in ε:

$$f(\eta,\phi,\zeta:\varepsilon) \sim \sum_{i=0}^{\infty} f^{i}(\eta,\phi,\zeta)\varepsilon^{i} \quad , \tag{29}$$

where $f^{i}(\eta,\phi,\zeta)$ are the boundary layer stress coefficients and $f^{i} = 0$ if $i < 0$. Substituting (28) and (29) into the equilibrium and compatibility equations (3) and (5) and equating coefficients of the same powers of ε, we obtain for the coefficients of ε^{n}:

$$\left\{\begin{aligned} & f^{n}_{z,\zeta} + f^{n}_{zx,\eta} + \sum{}'(f^{i}_{z} - f^{i}_{\theta}) + \sum{}'' f^{i}_{z\theta} = 0 \\ & f^{n}_{zx,\zeta} + f^{n}_{x,\eta} + \sum{}' f^{i}_{zx} + \sum{}'' f^{i}_{x\theta} = 0 \quad , \\ & f^{n}_{z\theta,\zeta} + f^{n}_{x\theta,\eta} + 2\sum{}' f^{i}_{z\theta} + \sum{}'' f^{i}_{\theta} = 0 \quad ; \end{aligned}\right. \tag{30}$$

$$\nabla^2 f_x^n + \Gamma^n_{,\eta\eta} + \overset{*}{\sum}{}' f_x^i = 0 \ ,$$

$$\nabla^2 f_z^n + \Gamma^n_{,\zeta\zeta} + \overset{*}{\sum}{}' f_z^i - 2 \sum{}''' (f_z^i - f_\Theta^i) + 4 \sum{}^{IV} f_{z\Theta}^i = 0,$$

$$\nabla^2 f_\Theta^n + \overset{*}{\sum}{}' [f_\Theta^i + \Gamma^i] + 2 \sum{}''' (f_z^i - f_\Theta^i) + 4 \sum{}^{IV} f_{z\Theta}^i = 0,$$

$$(31) \qquad \nabla^2 f_{zx}^n + \Gamma^n_{,\eta\zeta} + \overset{*}{\sum}{}' f_{zx}^i - \sum{}''' f_{zx}^i - 2 \sum{}^{IV} f_{x\Theta}^i = 0 \ ,$$

$$\nabla^2 f_{x\Theta}^n + \overset{*}{\sum}{}' f_{x\Theta}^i + \sum{}'' \Gamma^i_{,\eta} - \sum{}''' f_{x\Theta}^i + \sum{}^{IV} f_{zx}^i = 0 \ ,$$

$$\nabla^2 f_{z\Theta}^n + \overset{*}{\sum}{}' f_{z\Theta}^i + \sum{}'' \Gamma^i_{,\zeta} - 4 \sum{}''' f_{z\Theta}^i$$

$$+ \sum{}^{IV} [2(f_z^i - f_\Theta^i) - \Gamma^i] = 0 \ .$$

Here, the binomial expansions of $g(\zeta)$ and $g^2(\zeta)$ are employed, $\sum'$ and $\sum''$ are defined following Eq's. (11), and

$$\left.\begin{aligned}
\overset{*}{\sum}{}' A^i &\equiv \sum_{i+2(j+1)=n} \sum (-1)^j \zeta^j [A^i_{,\zeta} + (j+1) A^i_{,\phi\phi}] \ , \\
\sum{}''' A^i &\equiv \sum_{i+2(j+2)=n} \sum (-1)^j (j+1) \zeta^j A^i \ , \\
\sum{}^{IV} A^i &\equiv \sum_{i+2j+3=n} \sum (-1)^j (j+1) \zeta^j A^i_{,\phi} \ ,
\end{aligned}\right\} \quad i,\ j \geq 0 \ ,$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2} \ ,$$

$$\Gamma^n = \frac{1}{1+\nu} [f_x^n + f_z^n + f_\Theta^n] \ .$$

To complete the formulation, the boundary layer stresses must satisfy the boundary conditions at the ends of the cylinder and on the inner and outer surfaces. In addition these stresses should approach or "match" the interior stresses for "large" values of η, i.e., "small" values of ε.

From (6), (9) and (29) we find that on the boundaries of the cylinder the $f^n(\eta,\phi,\zeta)$ satisfy the following conditions:

$$f^n_{zx}(\eta,\phi,\pm 1) = f^n_{z\theta}(\eta,\phi,\pm 1) = 0 \ ,$$

(32)

$$\left\{f^n_z(\eta,\phi,1), f^n_z(\eta,\phi,-1)\right\} = \sum_{j=0}^{n-1} \left\{P^{n-j}_{oj},\ P^{n-j}_{Ij}\right\} \eta^j \ ;$$

$$f^n_x(0,\phi,\zeta) = \bar{\sigma}^n(\phi,\zeta) \ , \ f^n_{zx}(0,\phi,\zeta) = \bar{\tau}^n(\phi,\zeta) \ ,$$

(33)

$$f^n_{x\theta}(0,\phi,\zeta) = \bar{t}^n(\phi,\zeta) \ .$$

Here the functions $P^i_{oj}(\phi)$ and $P^i_{Ij}(\phi)$ are given by*:

$$P^i_{oj}(\phi) = \begin{cases} \dfrac{1}{j!}\dfrac{\partial p^i_o(0,\phi)}{\partial \xi^j} \ , & j = 0,1,\ldots, \\ 0 \ , & j < 0 \end{cases}$$

$$P^i_{Ij}(\phi) = \begin{cases} \dfrac{1}{j!}\dfrac{\partial p^i_I(0,\phi)}{\partial \xi^j} \ , & j = 0,1,\ldots, \\ 0 \ , & j < 0 \ . \end{cases}$$

*See (9a).

They are the coefficients obtained by expanding p_o^i and p_I^i in a Taylor series in ξ about $\xi = 0$. The formulas, (32), follow by substituting (28) in this series and then using (29).

The "matching conditions", or the asymptotic form of the boundary layer stress coefficients, for a fixed ϕ, are written as:

$$\lim_{\eta \to \infty} f^n(\eta,\phi,\zeta) = \overset{*}{\sigma}{}^n(\eta,\phi,\zeta), \qquad n = 0,1,\ldots, \tag{34}$$

where,

$$\overset{*}{\sigma}{}^n(\eta,\phi,\zeta) = \sum_{j=0}^{n} s_j^{n-j}(\phi,\zeta)\,\eta^j \quad , \tag{35a}$$

and

$$s_j^i(\phi,\zeta) = \begin{cases} \dfrac{1}{j!}\,\dfrac{\partial^j \sigma^i(0,\phi,\zeta)}{\partial \xi^j} \quad , & i \geq 0 \; , \\ 0 \quad , & i < 0 \; . \end{cases} \tag{35b}$$

The matching conditions are derived by assuming that in the neighborhood of $\xi = 0$ each interior stress coefficient can be expanded in a Taylor series in ξ. The conditions (34) and (35) follow* by substituting (28) into this expansion and using (8) and (29).

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*See Ref. 1 for details of the derivation.

5. Analysis of the Boundary Layer Problem.

In analyzing the boundary layer problem, Eq's. (30 - 35), we frequently use certain integral relations deduced from the solutions of Dirichlet problems for the harmonic and biharmonic equations on the semi-infinite strip, $D : |\zeta| \leq 1,\ \infty > \eta \geq 0$. These relations are given in the Appendix.

For every value of n equations (30 - 35) separate into two distinct systems which we denote as Problem P and Problem T. Problem P involves only f_x^n, f_z^n, f_Ω^n and f_{zx}^n, while Problem T considers the remaining two stress coefficients. For n = 0 we have for Problem P the first two of (30) and the first four of (31):

(36) $$f^o_{z,\zeta} + f^o_{zx,\eta} = 0\ , \qquad f^o_{zx,\zeta} + f^o_{x,\eta} = 0\ ;$$

(37) $$\begin{cases} \nabla^2 f^o_x + \Gamma^o_{,\eta\eta} = 0\ , & \nabla^2 f^o_z + \Gamma^o_{,\zeta\zeta} = 0\ , \\ \nabla^2 f^o_\Omega = 0\ , & \nabla^2 f^o_{zx} + \Gamma^o_{,\eta\zeta} = 0\ . \end{cases}$$

The appropriate boundary and matching conditions follow from (32 - 35):

(38) $$\begin{aligned} & f^o_{zx}(\eta,\phi,\pm 1) = f^o_z(\eta,\phi,\pm 1) = 0\ , \\ & f^o_x(0,\phi,\zeta) = \bar{\sigma}^o(\phi,\zeta),\ \ f^o_{zx}(0,\phi,\zeta) = \bar{\tau}^o(\phi,\zeta)\ ; \end{aligned}$$

$$\lim_{\eta \to \infty} \left\{ f^o_x, f^o_\theta, f^o_z, f^o_{zx} \right\} = \left\{ [S^o_x(0,\phi) + \bar{S}^o_x(0,\phi)\zeta], \right.$$

(39)

$$\left. [S^o_\theta(0,\phi) + \bar{S}^o_\theta(0,\phi)\zeta], [0], [0] \right\} .$$

In the boundary value problem (36 - 39) the variable ϕ is a parameter. This problem can be reduced to one for the biharmonic equation on D by introducing the reduced boundary layer stress coefficients, $F^o(\eta,\phi,\zeta)$, and an associated Airy stress function, $\chi^o(\eta,\phi,\zeta)$, in the following manner:

$$F^o_x \equiv f^o_x - \lim_{\eta \to \infty} f^o_x = \chi^o_{,\zeta\zeta}, \quad F^o_z \equiv f^o_z = \chi^o_{,\eta\eta},$$

(40)

$$F^o_{zx} \equiv f^o_{zx} = -\chi^o_{,\eta\zeta}, \quad F^o_\theta \equiv f^o_\theta - \lim_{\eta \to \infty} f^o_\theta = \nu\nabla^2\chi^o .$$

By direct substitution we see that (40) is a solution of (39 - 42) provided that χ^o is a solution for a fixed value of ϕ of the following boundary value problem on D:

$$\nabla^4 \chi^o = 0 ,$$

$$\chi^o_{,\eta\eta}(\eta,\phi, \pm 1) = \chi^o_{,\eta\zeta}(\eta,\phi, \pm 1) = 0 ,$$

$$\lim_{\eta \to \infty} [\chi^o_{,\eta\zeta}(\eta,\phi,\zeta), \chi^o_{,\zeta\zeta}(\eta,\phi,\zeta)] = 0 ,$$

(41)

$$\chi^o_{,\zeta\zeta}(0,\phi,\zeta) = \bar{\sigma}^o(\phi,\zeta) - S^o_x(0,\phi) - \bar{S}^o_x(0,\phi)\zeta ,$$

$$\chi^o_{,\eta\zeta} = -\bar{\tau}^o(\phi,\zeta) .$$

Application of integral relations I (see Appendix) to (41) yields,

(42a,b) $$S_x^o(0,\phi) = \frac{1}{2}\int_{-1}^{1}\bar{\sigma}^o(\phi,\zeta)d\zeta \, , \qquad \bar{S}_x^o(0,\phi) = \frac{3}{2}\int_{-1}^{1}\zeta\bar{\sigma}^o(\phi,\zeta)d\zeta \, ;$$

(42c) $$\int_{-1}^{1}\bar{\tau}^o(\phi,\zeta)d\zeta = 0 \quad .$$

Equations (42a,b) are the first two of the required four boundary conditions for the differential equations of the zeroth order interior problem, (24). Equation (42c) implies that $\bar{\tau}^o(\phi,\zeta)$ is a self-equilibrating force. For convenience we hereafter exclude all self-equilibrating edge forces and without loss of generality assume that $\bar{\sigma}^o(\phi,\zeta) \not\equiv 0$ and $\bar{\tau}^o(\phi,\zeta) \equiv 0$.

Problem T, for n = 0, is obtained from the remaining equilibrium and compatibility equations, boundary conditions and matching conditions in (30 - 35) with n = 0:

$$f^o_{z\Theta,\zeta} + f^o_{x\Theta,\eta} = 0 \; ; \qquad \nabla^2 f^o_{x\Theta} = \nabla^2 f^o_{z\Theta} = 0 \quad ;$$

(43) $$f^o_{z\Theta}(\eta,\phi,\pm 1) = 0 \; , \qquad f^o_{x\Theta}(0,\phi,\zeta) = \bar{t}^o(\phi,\zeta) \; ;$$

$$\lim_{\eta\to\infty}\left\{f^o_{z\Theta}, f^o_{x\Theta}\right\} = \left\{[0], [S^o_{x\Theta}(0,\phi) + \bar{S}^o_{x\Theta}(0,\phi)\zeta]\right\} .$$

Considering ϕ as a parameter the problem given by (43) can be converted into Neumann's problem for the harmonic equation on the semi-infinite strip. This is done by again introducing reduced boundary layer stress coefficients and an appropriate stress function $\psi^o(\eta,\phi,\zeta)$, such that

$$(44) \qquad F^{o}_{z\theta} \equiv f^{o}_{z\theta} = \psi^{o}_{,\zeta} \,, \qquad F^{o}_{x\theta} \equiv f^{o}_{x\theta} - \lim_{\eta \to \infty} f^{o}_{x\theta} = \psi^{o}_{,\eta} \;.$$

Equation (44) is a solution of (43) if ψ^{o} is a solution of the following boundary value problem on D:

$$\nabla^2 \psi^{o} = 0 \;,$$

$$\psi^{o}_{,\eta}(0,\phi,\zeta) = \bar{t}^{o}(\phi,\zeta) - S^{o}_{x\theta}(0,\phi) - \bar{S}^{o}_{x\theta}(0,\phi)\zeta \;,$$

$$(45)^{*} \qquad \psi^{o}_{,\zeta}(\eta,\phi,\pm 1) = 0 \;,$$

$$\lim_{\eta \to \infty} [\psi^{o}_{,\zeta},\ \psi^{o}_{,\eta}] = 0 \;.$$

Existence of a solution of (45) requires the integral of the normal derivative of ψ^{o} around the boundary to vanish. This yields,

$$(46) \qquad S^{o}_{x\theta}(0,\phi) = \frac{1}{2}\int_{-1}^{1} \bar{t}^{o}(\phi,\zeta)\,d\zeta \;,$$

which is the third boundary condition to be associated with (24). This completes the analysis of (30 - 35) for n = 0.

With n = 1 we obtain from (30 - 35), for Problem P, equations and boundary conditions identical, except for superscripts, with (36 - 38). The matching conditions are from (34), (35), (15b), (18) and (21),

---

*Instead of using the stress function ψ^{o} we could have employed a complex conjugate function. This leads to a Dirichlet problem for the harmonic equation.

$$\lim_{\eta \longrightarrow \infty} \left\{ f^1_x, f^1_\theta, f^1_z, f^1_{zx} \right\} = \left\{ [S^1_x(0,\phi) + \bar{S}^1_x(0,\phi)\zeta + (S^0_{x,\xi}(0,\phi) + \bar{S}^0_{x,\xi}(0,\phi)\zeta)\eta] , \right.$$

(47a)

$$[S^1_\theta(0,\phi) + \bar{S}^1_\theta(0,\phi)\zeta + (S^0_{\theta,\xi}(0,\phi) + \bar{S}^0_{\theta,\xi}(0,\phi)\zeta)\eta], [0] ,$$

$$\left. [\tfrac{1}{2}(\bar{S}^0_{x,\xi}(0,\phi) + \bar{S}^0_{x\theta,\phi}(0,\phi))(1 - \zeta^2)] \right\} .$$

We again introduce reduced boundary layer stress coefficients, $F^1(\eta,\phi,\zeta)$ and an associated stress function $\chi^1(\eta,\phi,\zeta)$ through the relations:

$$F^1_x = f^1_x - \lim_{\eta \longrightarrow \infty} f^1_x = \chi^1_{,\zeta\zeta} + \zeta\psi^0_{,\phi\zeta}$$

$$F^1_z = f^1_z = \chi^1_{,\eta\eta} - \zeta\psi^0_{,\phi\zeta} ,$$

(47b)

$$F^1_{zx} = f^1_{zx} - \lim_{\eta \longrightarrow \infty} f^1_{zx} = -\chi^1_{,\eta\zeta} - \zeta\psi^0_{,\phi\eta} ,$$

$$F^1_\theta = f^1_\theta - \lim_{\eta \longrightarrow \infty} f^1_\theta = \nu\nabla^2\chi^1 + 2(1 + \nu)\psi^0_{,\phi} .$$

Equations (47) provide a solution of the equations and boundary and matching conditions of Problem P with n = 1 if χ^1 is a solution, for every ϕ, of the following boundary value problem on the semi-infinite strip:

$$\nabla^4 \chi^1 = 0 \quad ,$$

$$\chi^1_{,\eta\eta}(\eta,\phi,\pm 1) = 0, \chi^1_{,\eta\zeta}(\eta,\phi,\pm 1) = \mp \psi^0_{;\phi\eta}(\eta,\phi,\pm 1) \; ,$$

$$(48) \quad \lim_{\eta\to\infty} [\chi^1_{,\eta\zeta}(\eta,\phi,\zeta), \chi^1_{,\zeta\zeta}(\eta,\phi,\zeta)] = 0 \; ,$$

$$\chi^1_{,\zeta\zeta}(0,\phi,\zeta) = \bar{\sigma}^1(\phi,\zeta) - S^1_x(0,\phi) - \bar{S}^1_x(0,\phi)\zeta - \zeta\psi^0_{;\phi\zeta}(0,\phi,\zeta) \; ,$$

$$\chi^1_{,\eta\zeta}(0,\phi,\zeta) = -\bar{\tau}^1(\phi,\zeta) + \tfrac{1}{2}[\bar{S}^0_{x,\xi}(0,\phi) + \bar{S}^0_{x\theta,\phi}(0,\phi)](1-\zeta^2) - \zeta\psi^0_{;\phi\eta} \; .$$

Employing (45), the third of the integral relations I (see Appendix) applied to (48) yields the fourth and final boundary condition for the zeroth order interior problem, (24):

$$(49) \quad \tfrac{2}{3}[2\bar{S}^0_{x\theta,\phi}(0,\phi) + \bar{S}^0_{x,\xi}(0,\phi)] = \int_{-1}^{1} \bar{\tau}^1(\phi,\zeta)d\zeta + \int_{-1}^{1} \zeta\bar{t}^0_{;\phi}(\phi,\zeta)d\zeta \; .$$

The boundary conditions, (42a,b), (46) and (49) coincide with those usually associated with the Donnell equations.[5] Equations (42a) and (46) correspond to the prescription of the resultant axial and twist forces respectively. While (42b) and (49) are respectively the boundary conditions for the resultant bending moment and "shear reaction."

The remaining two integral relations applied to (48) give the following boundary conditions for the first order interior differential equations (27),

$$S^1_x(0,\phi) = \tfrac{1}{2}\int_{-1}^{1}\bar{\sigma}^1(\phi,\zeta)d\zeta + \tfrac{1}{2}\int_{-1}^{1}\psi^o_{;\phi}(0,\phi,\zeta)d\zeta \quad ,$$

(50)

$$\bar{S}^1_x(0,\phi) = \tfrac{3}{2}\int_{-1}^{1}\zeta\bar{\sigma}^1(\phi,\zeta)d\zeta + 3\int_{-1}^{1}\zeta\psi^o_{;\phi}(0,\phi,\zeta)d\zeta \;.$$

The remaining equations and conditions contained in (30 - 35) for n = 1 yield the following Problem T:

$$f^1_{z\theta,\zeta} + f^1_{x\theta,\eta} = -f^o_{\theta,\phi}, \quad \nabla^2 f^1_{x\theta} = -\Gamma^o_{;\phi\eta},$$

$$\nabla^2 f^1_{z\theta} = -\Gamma^o_{;\phi\zeta} \;;$$

(51) $$f^1_{z\theta}(\eta,\phi,\pm 1) = 0 \;, \quad f^1_{x\theta}(0,\phi,\zeta) = \bar{t}^1(\phi,\zeta) \;;$$

$$\lim_{\eta\to\infty}\left\{f^1_{z\theta}, f^1_{x\theta}\right\} = \Big\{[\tfrac{1}{2}(\bar{S}^o_{\theta,\phi}(0,\phi) + \bar{S}^o_{x\theta,\xi}(0,\phi))(1-\zeta^2)] \;,$$

$$[S^1_{x\theta}(0,\phi) + \bar{S}^1_{x\theta}(0,\phi)\zeta + (S^o_{x\theta,\xi}(0,\phi) + \bar{S}^o_{x\theta,\xi}(0,\phi)\zeta)\eta]\Big\}.$$

The boundary value problem given by (51) can be reduced to one for Poisson's equation on the semi-infinite strip by defining reduced boundary layer stress coefficients and a stress function, $\Xi^1(\eta,\phi,\zeta)$, such that,

$$F^1_{z\theta} \equiv f^1_{z\theta} - \lim_{\eta\to\infty} f^1_{z\theta} = \Xi^1_{,\eta} - \nu\chi^o_{;\phi\zeta} \quad ,$$

(52)

$$F^1_{x\theta} \equiv f^1_{x\theta} - \lim_{\eta\to\infty} f^1_{x\theta} = -\Xi^1_{,\zeta} - \nu\chi^o_{;\phi\eta} \;.$$

Here we have used (40) and (41). Equations (52) give a solution of (51) if Ξ^1 is a solution of the following boundary value problem on D:

$$\nabla^2 \Xi^1 = (1 - \nu)\int_\eta^\infty \nabla^2 \chi^0_{,\phi\zeta} d\eta \ ,$$

$$(53) \qquad \Xi^1_{,\eta}(\eta,\phi, \pm 1) = 0, \ \lim_{\eta \to \infty}\left[\Xi^1_{,\eta}, \quad \Xi^1_{,\zeta}\right] = 0 \ ,$$

$$\Xi^1_{,\zeta}(0,\phi,\zeta) = -\bar{t}^1(\phi,\zeta) + S^1_{x\Omega}(0,\phi) + \bar{S}^1_{x\Omega}(0,\phi)\zeta \ ,$$

where we have employed (41) and the previous assumption that $\bar{\tau}^0(\phi,\zeta) \equiv 0$. Application of the integral relation II of the Appendix to (53) gives the third boundary condition for the first order interior problem as:

$$(54) \qquad S^1_{x\Omega}(0,\phi) = \frac{1}{2}\int_{-1}^{1} \bar{t}^1(\phi,\zeta)d\zeta \ .$$

This completes the analysis of (30 - 35) for n = 1.

The remaining boundary condition for the first order interior problem and boundary value problems for determining the boundary layer stress coefficients of order two are obtained from an analysis of (30 - 35) with n = 2. This analysis is lengthy and somewhat involved and is therefore not presented. We merely list the result of applying the third of the integral relations I. This gives the fourth and final boundary condition for the first order interior problem as:

$$\frac{2}{3}[2\bar{S}^1_{x\theta,\phi}(0,\phi) + \bar{S}^1_{x,\xi}(0,\phi)] = \int_{-1}^{1} \bar{\tau}^2(\phi,\zeta)d\zeta + \int_{-1}^{1} \bar{t}^1_{,\phi}(\phi,\zeta)d\zeta \ . \tag{55}$$

Higher order approximations may be obtained by examining (30 - 35) with $n > 2$.

6. A Summary of the Shell Theory.

In the previous sections we have obtained by a systematic expansion procedure in the "small" parameter ε, a sequence of boundary value problems whose solutions approximate the three dimensional state of stress in a cylindrical shell. The approximation can be carried to any desired degree. The basic element of the approximation is a thin shell theory which forms part of the zeroth order interior problem. The proper boundary conditions for this theory are obtained from the boundary layer analysis. It is implied by the expansion procedure that the higher order approximations serve as "corrections" to the basic thin shell approximation yielding "thick shell" theories of increasing degrees of accuracy. We therefore define the <u>thick shell theory of order N</u> through the relations:

$$\begin{aligned}\left\{\sigma_x^{(N)},\sigma_\theta^{(N)},\sigma_{x\theta}^{(N)}\right\} = \sum_{i=0}^{N}\Big[&\left\{\sigma_x^i(\xi,\phi,\zeta),\sigma_\theta^i(\xi,\phi,\zeta),\sigma_{x\theta}^i(\xi,\phi,\zeta)\right\} \\ &+ \left\{F_x^i(\xi/\varepsilon,\phi,\zeta),F_\theta^i(\xi/\varepsilon,\phi,\zeta),F_{x\theta}^i(\xi/\varepsilon,\phi,\zeta)\right\}\Big]\varepsilon^i \quad ,\end{aligned} \tag{56a}$$

$$\begin{aligned}\left\{\sigma_{zx}^{(N)},\sigma_{z\theta}^{(N)}\right\} &= \sum_{i=1}^{N+1}\left\{\sigma_{zx}^i(\xi,\phi,\zeta),\sigma_{z\theta}^i(\xi,\phi,\zeta)\right\}\varepsilon^i \\ &+ \sum_{i=0}^{N}\left\{F_{zx}^i(\xi/\varepsilon,\phi,\zeta) + F_{z\theta}^i(\xi/\varepsilon,\phi,\zeta)\right\}\varepsilon^i \quad ,\end{aligned} \tag{56b}$$

(56c) $$\sigma_z^{(N)} = \sum_{i=2}^{N+2} \sigma_z^i(\xi,\phi,\zeta)\varepsilon^i + \sum_{i=0}^{N} F_z^i(\xi/\varepsilon,\phi,\zeta) \quad .$$

Here, $\sigma^{(N)}$ are the stresses associated with the N^{th} order thick shell theory, and σ^i and F^i are respectively the interior and the reduced boundary layer stress coefficients of i^{th} order.

In summary we list below the boundary value problems from which the stresses of the zeroth and first order thick shell theory may be computed.

Zeroth order interior problem:

Differential equations: Equations (24);

Stress displacement relations: Equations (25), (26) and (19);

Boundary conditions, on $\xi = 0$ (similar ones apply to $\xi = L'$): (42a,b), (46) and (49).

The differential equations of the zeroth order interior problem can be formulated in a somewhat different but equivalent manner by introducing an Airy stress function $G^o(\xi,\phi)$ such that,

(57) $$S_x^o = G^o_{,\phi\phi}, \quad S_\theta^o = G^o_{,\xi\xi}, \quad S_{x\theta}^o = -G^o_{,\phi\xi} \quad .$$

Employing (57) we find that the first two of (24) are identically satisfied, see also (20). Eliminating U^1 and V^1 from the first three of (19a) with $n = 0$ we find, using (57), that

(58a) $$\Delta^2 G^o - W^o_{,\xi\xi} = 0 \quad ,$$

and that the third of (24) becomes*,

$$\Delta^2 W^o + 3(1-\nu^2)G^o_{,\xi\xi} = 3(1-\nu^2)(p_o^2 - p_I^2) \tag{58b}$$

The differential equations (24) can therefore be replaced by those in (58). Appropriate boundary conditions for (58a) are obtained from (57), (42a) and (46).

Zeroth order boundary layer problems:

Boundary value problems: Equations (41) and (45);

Stress relations: Equations (39), (40), (43) and (44).

First order interior problem:

The differential equations and the stress displacement relations are identical with those of the zeroth order interior problem provided a, 1, is added to the superscript of each term. The boundary conditions on $\xi = 0$ are given by (50), (54) and (55). Similar conditions apply to $\xi = L'$.

First order boundary layer problems:

Boundary value problems: Equations (48) and (53);

Stress relations: Equations (47), (51) and (52).

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*By introducing the complex function, $H^o(\xi,\phi) = W^o(\xi,\phi) + i\sqrt{3(1-\nu^2)}G^o(\xi,\phi)$, the system (59) may be combined into the single complex equation,

$$\Delta^2 H^o - i\sqrt{3(i-\nu^2)}H^o_{,\xi\xi} = 3(1-\nu^2)(p_o^2 - p_I^2).$$

In applying the thick shell theory to specific problems it is important to observe the order of solution. The zeroth order interior problem must be solved first. The solution thus obtained yields inhomogeneous terms for the zeroth order boundary layer problems. The solutions of these problems in turn provides inhomogeneous terms for the first order interior problem and so forth.

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Appendix

Consider the following boundary value problem for $\chi(\eta,\zeta)$ defined on D:

$$\nabla^4 \chi = 0 \quad ,$$

$$\chi,_{\zeta\zeta}(0,\zeta) = f(\zeta), \quad \chi,_{\eta\zeta}(0,\zeta) = g(\zeta), \quad \chi,_{\eta\eta}(\eta, \pm 1) = 0 \ ,$$

$$\lim_{\eta \longrightarrow \infty} [\chi,_{\zeta\zeta}(\eta,\zeta), \quad \chi,_{\eta\zeta}(\eta,\zeta)] = 0,$$

$$\chi,_{\eta\zeta}(\eta,1) = k_1(\eta),$$

$$\chi,_{\eta\zeta}(\eta,-1) = k_2(\eta).$$

It is easy to show that if $\chi,_{\zeta\zeta}$, $\chi,_{\eta\zeta}$ and $\chi,_{\zeta}$ are continuous functions and if χ, $\chi,_{\eta}$ and $\chi,_{\zeta}$ are single valued functions then,

$$\int_{-1}^{1} f(\zeta)d\zeta = \int_{0}^{\infty} [k_2(\eta)-k_1(\eta)]d\eta \ ,$$

I $$\int_{-1}^{1} \zeta f(\zeta)d\zeta = -\int_{0}^{\infty} [k_1(\eta) + k_2(\eta)]d\eta \ ,$$

$$\int_{-1}^{1} g(\zeta)d\zeta = 0 \quad .$$

Consider the following boundary value problem for $\Xi(\eta,\zeta)$ defined on the semi-infinite strip:

$$\nabla^2 \Xi = 0 \ ,$$

$$\Xi_{,\zeta}(0,\zeta) = f(\zeta) \ , \qquad \Xi_{,\eta}(\eta, \pm 1) = 0 \ ,$$

$$\lim_{\eta \to \infty} \left[\Xi_{,\eta}(\eta,\zeta) \ , \qquad \Xi_{,\zeta}(\eta,\zeta) \right] = 0 \ .$$

It is easy to show that if $\Xi_{,\zeta}$ is a continuous function and if Ξ is a single valued function then,

II $$\int_{-1}^{1} f(\zeta) d\zeta = 0 \ .$$

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